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THE DARWIN MODEL AS A TOOL FOR ELECTROMAGNETIC
PLASMA SIMULATION*

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ABSTRACT

The Darwin model of electromagnetic interaction is presented as a self-consistent theory, and is shown to be an excellent approximation to the Maxwell theory for slow electromagnetic waves. Since the fast waves of the Maxwell theory are absent, it is convenient for use in the computer simulation of the electromagnetic dynamics of nonrelativistic plasma.

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Simulation studies of plasma behavior have usually been based either on the Coulomb model of particle interaction or on the full Maxwell theory. For the consideration of electromagnetic effects, it is not necessary to use the Maxwell theory, which includes the propagation of fast waves (whose group velocities are of the order of c), and which thus has the disadvantage of requiring a small time step. Indeed, the many plasma phenomena in which fast waves are unimportant are well described by the Darwin model¹ of electrodynamics.

The Darwin model, in which there is no retardation, has been used in the past to study electromagnetic interactions in microscopic systems,² and for the statistical mechanics of many-body systems, both neutral³ and plasma.⁴ While for nonequilibrium plasma the Darwin model has been little used in analytic theoretical work, it seems most suitable for simulation studies, as pointed out by Hasagawa and Okuda,⁵ who rediscovered it in simulating one dimensional plasma dynamics

In this paper we investigate the analytic properties of the Darwin model. The model is usually treated as a second order (in v/c) approximation to the relativistic Maxwell theory.⁶ We present it here as a self consistent theory arising from a particle Lagrangian:

$L = \sum \frac{1}{2} m_i v_i^2 - C + M$, where the kinetic energy is nonrelativistic, while the Coulomb energy, $C \equiv \frac{1}{2} \sum e_i \phi^i + \sum e_i \phi^{ex}(\underline{r}_i, t)$, and the magnetic energy $M \equiv \frac{1}{2} \sum e_i (\underline{v}_i/c) \cdot \underline{A}^i + \sum e_i (\underline{v}_i/c) \cdot \underline{A}^{ex}(\underline{r}_i, t)$, include both pair interactions and interactions with external potentials. The internal potentials are defined as

$$\phi^i(\underline{r}_i, \{\underline{r}_j\}) \equiv \sum_{j(\neq i)} e_j / r_{ij}, \quad (1a)$$

$$\tilde{A}^i(\tilde{r}_i, \{\tilde{r}_j\}) \equiv \sum_{j(\neq i)} e_j \tilde{T}(\tilde{r}_{ij}) \cdot (\tilde{v}_j/c). \quad (1b)$$

The magnetic interaction tensor is $\tilde{T}(\tilde{r}) \equiv \frac{1}{2} (\tilde{I} + \hat{r}\hat{r})/r$. The internal potential (1b) is solenoidal because $(\partial/\partial\tilde{r}) \cdot \tilde{T}(\tilde{r}) = 0$. We therefore choose the transverse or Coulomb gauge also for the external potential: $\nabla \cdot \tilde{A}^{\text{ex}} = 0$.

The canonical momentum of a particle

$$\tilde{p}_i \equiv \partial L / \partial \tilde{v}_i = m_i \tilde{v}_i + (e_i/c) [\tilde{A}^i + \tilde{A}^{\text{ex}}(\tilde{r}_i, t)] \quad (2)$$

is a function of the velocities of all the particles through \tilde{A}^i . The Lagrangian equation $\dot{\tilde{r}}_i = \partial L / \partial \tilde{p}_i$ is the standard Newton-Lorentz equation of motion, when the electromagnetic field is expressed in terms of the potentials.

When the external potentials are static, the invariance of L under time-translation implies conservation of energy $dH/dt = 0$, where the Hamiltonian $H \equiv \sum \tilde{p}_i \cdot \tilde{v}_i - L$ has a simple form $H = \sum \frac{1}{2} m_i \tilde{v}_i^2 + C + M$ in terms of velocities (but not in terms of momenta^{3,4}). When the external field vanishes, the invariance of L under space-translation implies conservation of total canonical momentum: $(d/dt) \sum \tilde{p}_i(t) = 0$.

If the set of discrete particles is approximated by a continuum with charge and current densities $\rho(\tilde{r}, t)$ and $\tilde{j}(\tilde{r}, t)$, the internal potentials $\{\phi^i, \tilde{A}^i\}$ become fields $\phi^{\text{in}}(\tilde{r}, t)$, $\tilde{A}^{\text{in}}(\tilde{r}, t)$ satisfying

$$-\nabla^2 \phi^{\text{in}} = 4\pi \rho, \quad (3a)$$

$$-\nabla^2 \tilde{A}^{\text{in}} = 4\pi \tilde{j}_T/c, \quad \nabla \cdot \tilde{A}^{\text{in}} = 0. \quad (3b)$$

The source of \tilde{A}^{in} is the transverse (or solenoidal) part of \tilde{j} , defined most simply in terms of spatial Fourier transforms:

$\tilde{j}_T(k, t) \equiv (\mathbb{I} - \hat{k}\hat{k}) \cdot \tilde{j}(k, t)$. Note that the retardationless elliptic equation (3b) has the same form as the Poisson equation (3a), whose solution is standard in simulation studies.

Of the four Darwin equations for the internal electromagnetic field, only one:

$$c\nabla \times \tilde{B}^{in} = 4\pi\tilde{j} + \partial\tilde{E}_L^{in}/\partial t \quad (4)$$

differs from Maxwell's; the others are the same. Here $\tilde{E}_L \equiv -\nabla\phi$ is the longitudinal (or irrotational) part of the electric field. The longitudinal part of (4), $0 = 4\pi\tilde{j}_L + \partial\tilde{E}_L^{in}/\partial t$, expresses charge conservation in agreement with (3a). The transverse part of (4), $c\nabla \times \tilde{B}^{in} = 4\pi\tilde{j}_T$, differs from Maxwell theory in omitting $\partial\tilde{E}_T^{in}/\partial t$, and leads to the Biot-Savart law: $\tilde{B}^{in}(\tilde{r}, t) = c^{-1} \int d^3r' (\nabla|\tilde{r} - \tilde{r}'|^{-1}) \times \tilde{j}(\tilde{r}', t)$.

The differential conservation laws differ from those of the Maxwell theory. If we require the external field also to satisfy the Darwin field equations, the energy equation is

$$\begin{aligned} -\tilde{E} \cdot \tilde{j}^{ex} = & (\partial/\partial t) \left[(|\tilde{E}_L|^2 + B^2)/8\pi + \sum_s \int d^3v \frac{1}{2} m_s v^2 f_s \right] + \nabla \cdot \left[(c/4\pi) \tilde{E} \times \tilde{B} \right. \\ & \left. + (4\pi c)^{-1} (\partial\tilde{A}/\partial t) (\partial\phi/\partial t) + \sum_s \int d^3v \frac{1}{2} m_s v^2 \tilde{v} f_s \right], \end{aligned}$$

where $f_s(\tilde{r}, \tilde{v}, t)$ is the particle distribution function, satisfying the standard Vlasov equation. The momentum equation is, in the absence of external fields, $(\partial/\partial t)(c^{-1}\rho\tilde{A} + \sum_s \int d^3v m_s \tilde{v} f_s) = -\nabla \cdot \tilde{P}$, where the momentum flux density is $\tilde{P} \equiv \sum_s \int d^3v m_s \tilde{v} \tilde{v} f_s + \mathbb{I}(B^2 + |\tilde{E}_L|^2 + 2\tilde{A} \cdot \tilde{E}_L)/8\pi$

$$- \left[\underline{\underline{B}}\underline{\underline{B}} + \underline{\underline{E}}_L \underline{\underline{E}}_L + c^{-1} (\underline{\underline{A}} \dot{\underline{\underline{E}}}_L + \dot{\underline{\underline{E}}}_L \underline{\underline{A}}) \right] / 4\pi. \quad (\text{Here } \dot{\underline{\underline{E}}}_L \equiv \partial \underline{\underline{E}}_L / \partial t.)$$

To see how the Darwin model modifies the plasma dynamics, we consider linear oscillations of a uniform plasma. With $\underline{\underline{j}}(\underline{\underline{k}}, \omega) = \underline{\underline{g}}(\underline{\underline{k}}, \omega) \cdot \underline{\underline{E}}(\underline{\underline{k}}, \omega)$ and the Darwin field equations, we obtain

$$\left[\underline{\underline{\epsilon}}(\underline{\underline{k}}, \omega) - (n^2 + 1) \underline{\underline{I}}_T \right] \cdot \underline{\underline{E}}(\underline{\underline{k}}, \omega) = 0, \quad (5)$$

where $\underline{\underline{\epsilon}} \equiv \underline{\underline{I}} + 4\pi i \omega^{-1} \underline{\underline{g}}$, $n^2 \equiv k^2 c^2 / \omega^2$, $\underline{\underline{I}}_T \equiv \underline{\underline{I}} - \hat{\underline{\underline{k}}} \hat{\underline{\underline{k}}}$, and $\underline{\underline{g}}$ must be derived from the linearized kinetic equation. The dispersion equation $\omega(\underline{\underline{k}})$ results from the vanishing of the determinant of (5). The Maxwell equations, on the other hand, yield

$$\left[\underline{\underline{\epsilon}}(\underline{\underline{k}}, \omega) - n^2 \underline{\underline{I}}_T \right] \cdot \underline{\underline{E}}(\underline{\underline{k}}, \omega) = 0. \quad (6)$$

For a cold plasma in a static magnetic field, $\underline{\underline{\epsilon}}$ is independent of $\underline{\underline{k}}$. Then a simple relation

$$n_D^2(\omega, \theta) = n_M^2(\omega, \theta) - 1 \quad (7)$$

exists between the Darwin and Maxwell refractive indices n_D and n_M . (θ is the angle between $\underline{\underline{k}}$ and $\underline{\underline{B}}^{\text{ex}}$.) Curves of $n_M^2(\omega)$, for $\theta = 0$, $\theta = \pi/2$, and $0 < \theta < \pi/2$, may be found, e.g., in Refs. 7 and 8. From these diagrams (and from the associated analysis), we see that for frequencies less than the highest resonant ($n^2 = \infty$) frequency, $|n_M^2|$ is generally large compared to one, i.e., the phase velocity is much less than c . The exceptions are the small frequency intervals near the cutoffs ($n^2 = 0$), where $|n^2| \lesssim 1$. We conclude that, except near the cutoffs, the Darwin model agrees well with the Maxwell results. For frequencies above the highest resonance, $n_M^2 < 1$ and therefore

$n_D^2 < 0$. The supraluminous waves of the Maxwell theory, which are troublesome in plasma simulation, are replaced by evanescent waves in the Darwin model. In terms of an ω vs k diagram, the fast branch ($\omega \rightarrow kc$ as $k \rightarrow \infty$) simply disappears.

For a hot plasma, where ϵ depends also on k , no simple relation like (7) can be found. However, comparison of (6) with (5) again leads us to conclude that the Darwin model is reliable for slow waves ($n^2 \gg 1$) and unreliable for fast waves ($n^2 \lesssim 1$).

Finally, for an inhomogeneous plasma, the new modes (drift waves) have still lower frequencies and phase velocities, and hence the Darwin model should be valid here as well.

We conclude that, in plasma simulation work, it is not necessary to choose between a Coulomb model and the full Maxwell theory; the Darwin model presents a third intermediate possibility. It describes electric and magnetic forces, includes induced fields as well as static fields, and yet retains much of the simple structure of the Coulomb model. Since the electrodynamics of a nonrelativistic plasma usually deals with slow waves, it is seen that the Darwin model has a wide range of applicability.

In this discussion, we have made no attempt to deal with the formulation of differencing schemes, nor to face the difficulties pointed out by Hasegawa and Okuda concerning boundary conditions.⁵

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FOOTNOTES AND REFERENCES

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